COMS BC1016

Introduction to Computational Thinking and Data Science

Lecture 16: Normal Distribution

Nov 12, 2025

Reminders

- Final Project Proposal Due ~~Friday~~ **Wednesday, Nov 19**
    - Worth **10%** of the final project grade
    - Template is on the 1017 Courseworks
- HW 6 due Monday, Nov 17
- HW 7: Skip Question 4 about the survey
- Extra Credit (HW 5 Question 3) due Monday, Nov 17
    - Completely optional, no late submissions

Lecture Outline

- Summary of Hypothesis Testing
- Normal Distributions
  - Standard Deviation
  - Standard Units
  - Central Limit Theorem

Data Science in this course

- Exploration: Discover patterns in data and articulate insights (visualizations)

- Inference: Make reliable conclusions about the world

  - Statistics is useful

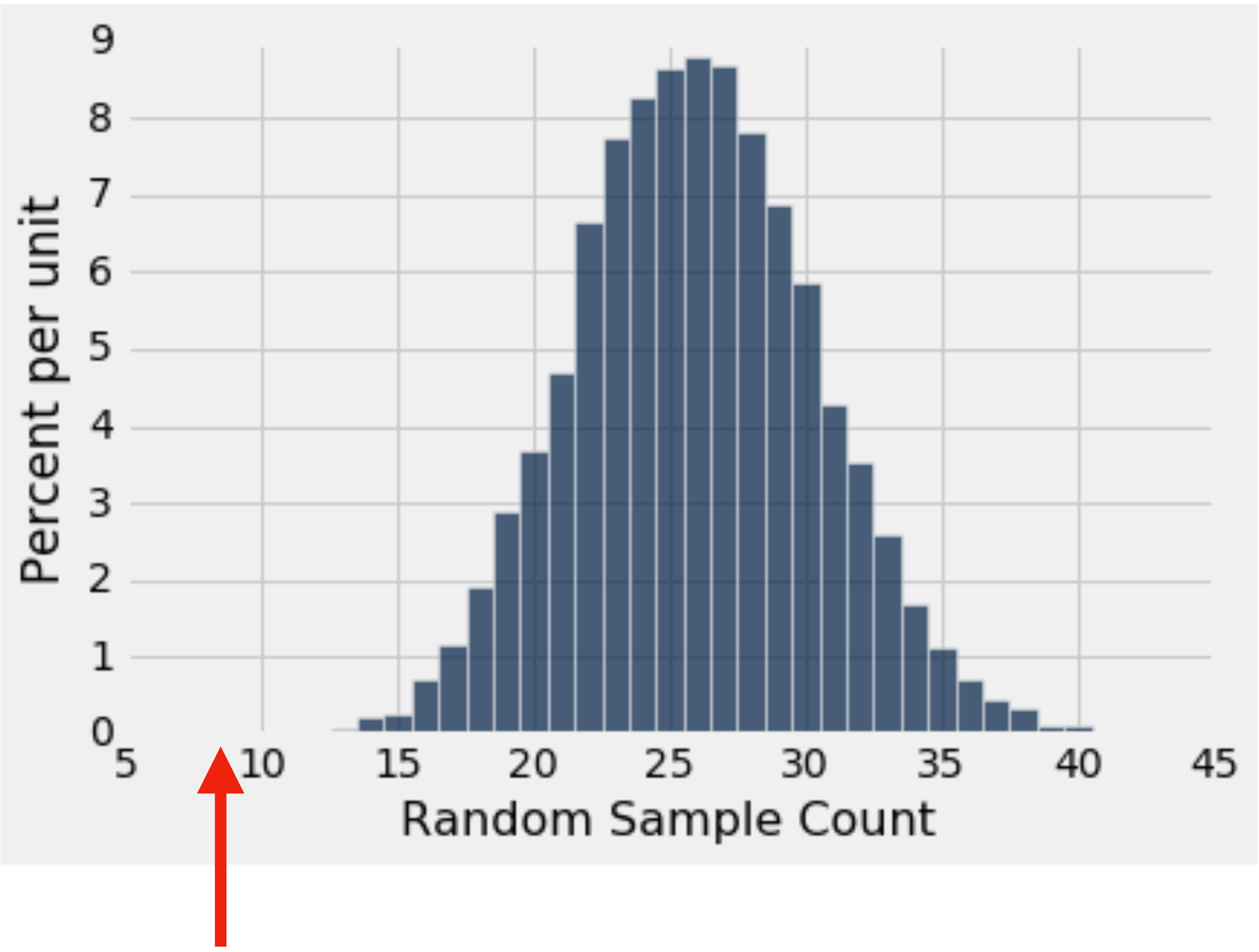
- **Prediction: Informed guesses about unseen data**

Summary of stats so far…

Hypothesis Testing

- Modeling expected outcomes under the null and comparing it to our observed outcome

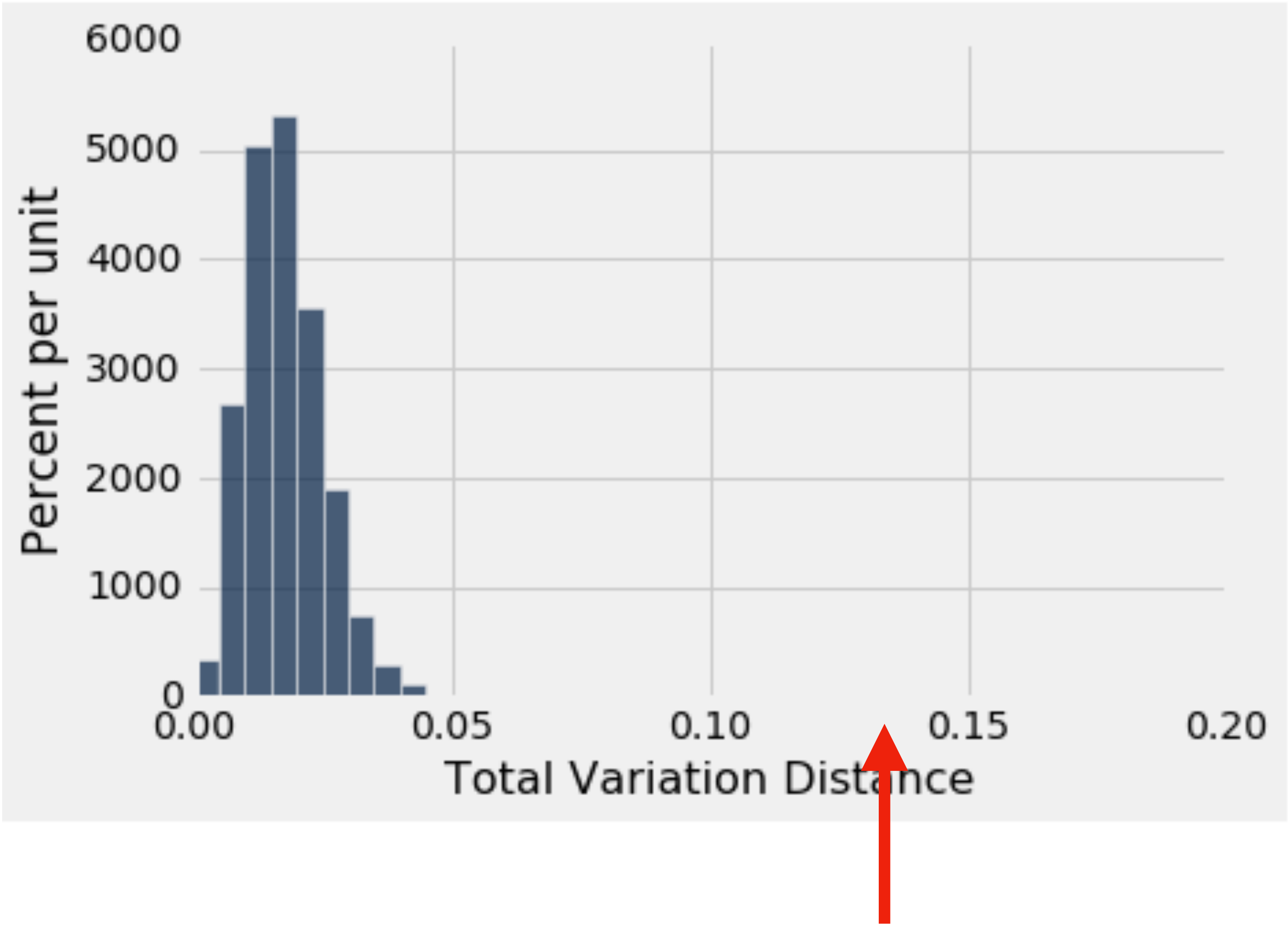
Swain v Alabama



Observed Number

2 Categories

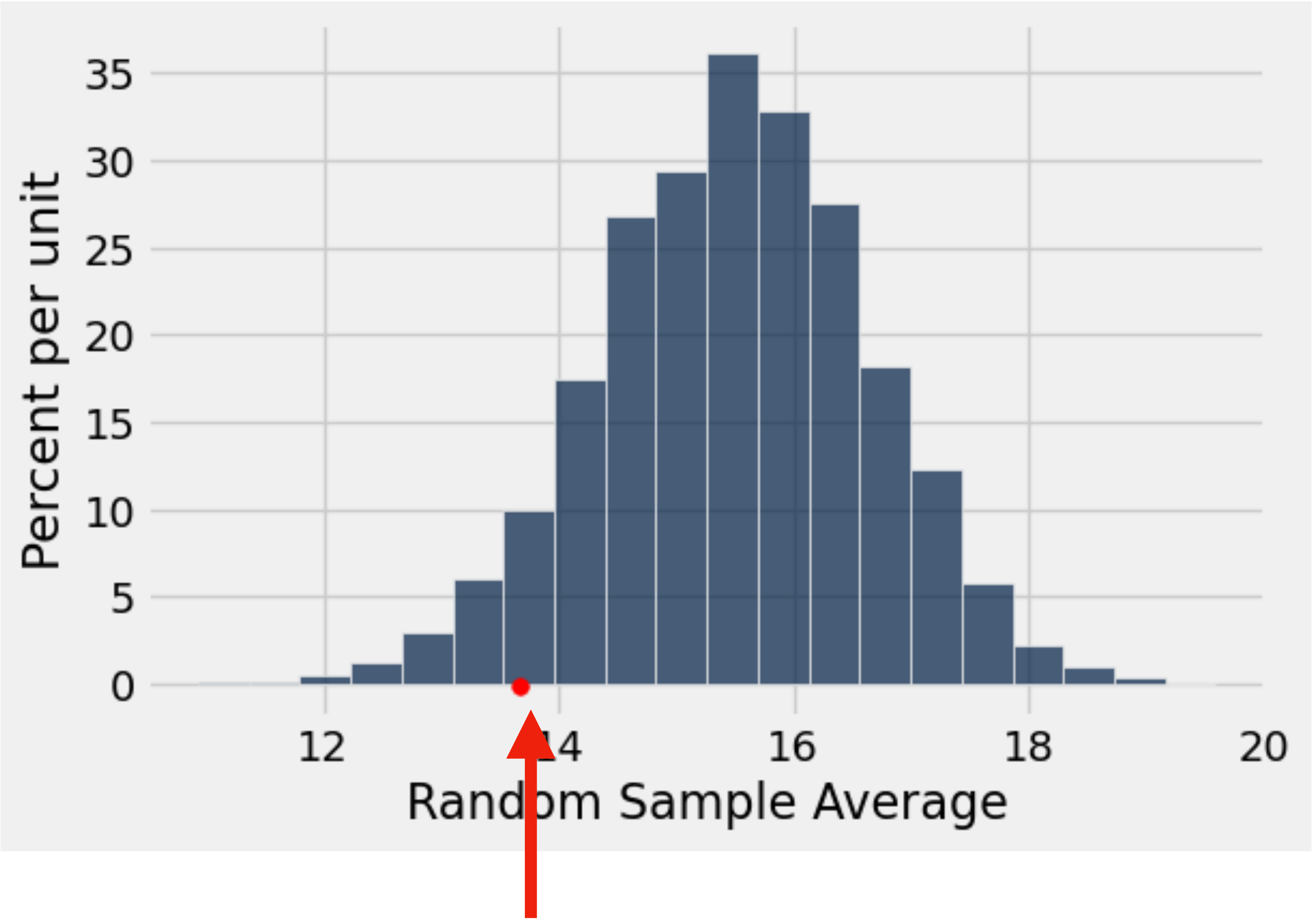
Alameda Jury



Observed TVD

3+ Categories

Midterm Exam Scores



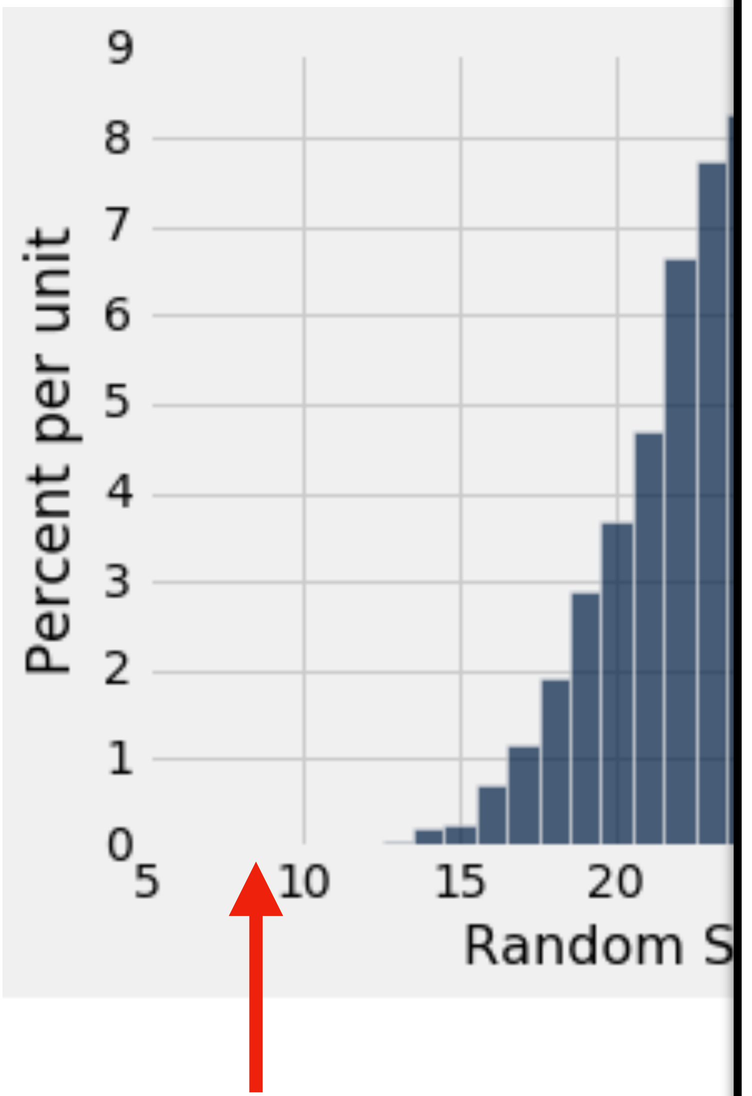
Observed Average

Numerical Data

Hypothesis Testing

- Modeling expected outcomes under the null and comparing it to our observed outcome

Swain v



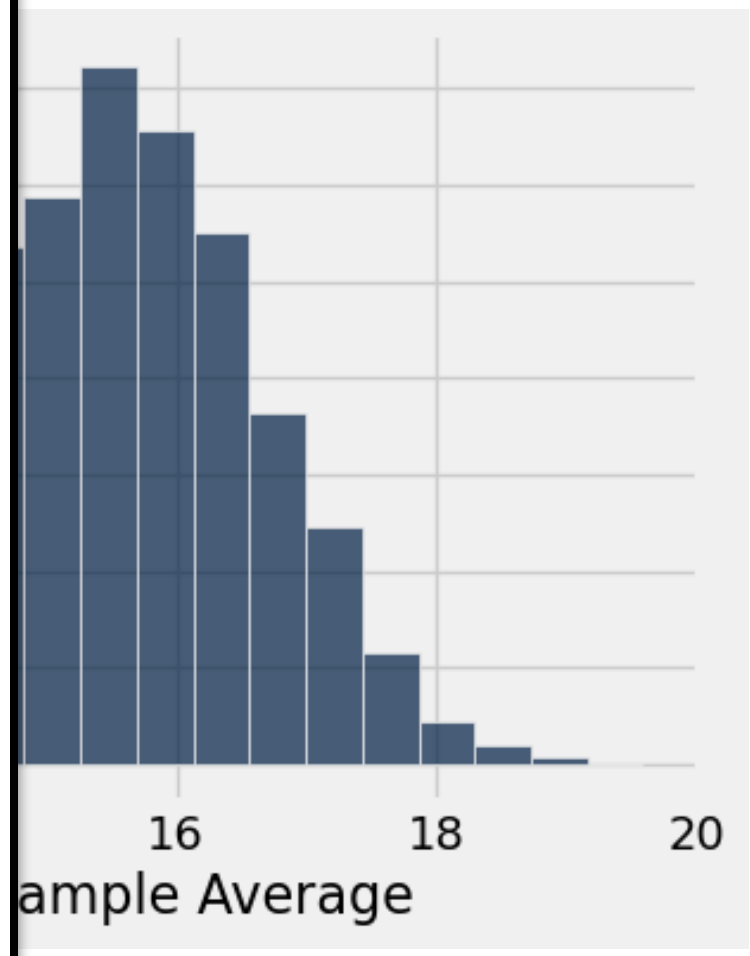
Observed N

It's often not easy to say whether the observed outcome falls within our expectations…

How can we more precisely characterize the likelihood of observing an expected outcome?

p-value!

xam Scores



Average

2 Categories

3+ Categories

Numerical Data

Hypothesis Testing

- Modeling expected outcomes under the null and comparing it to our observed outcome

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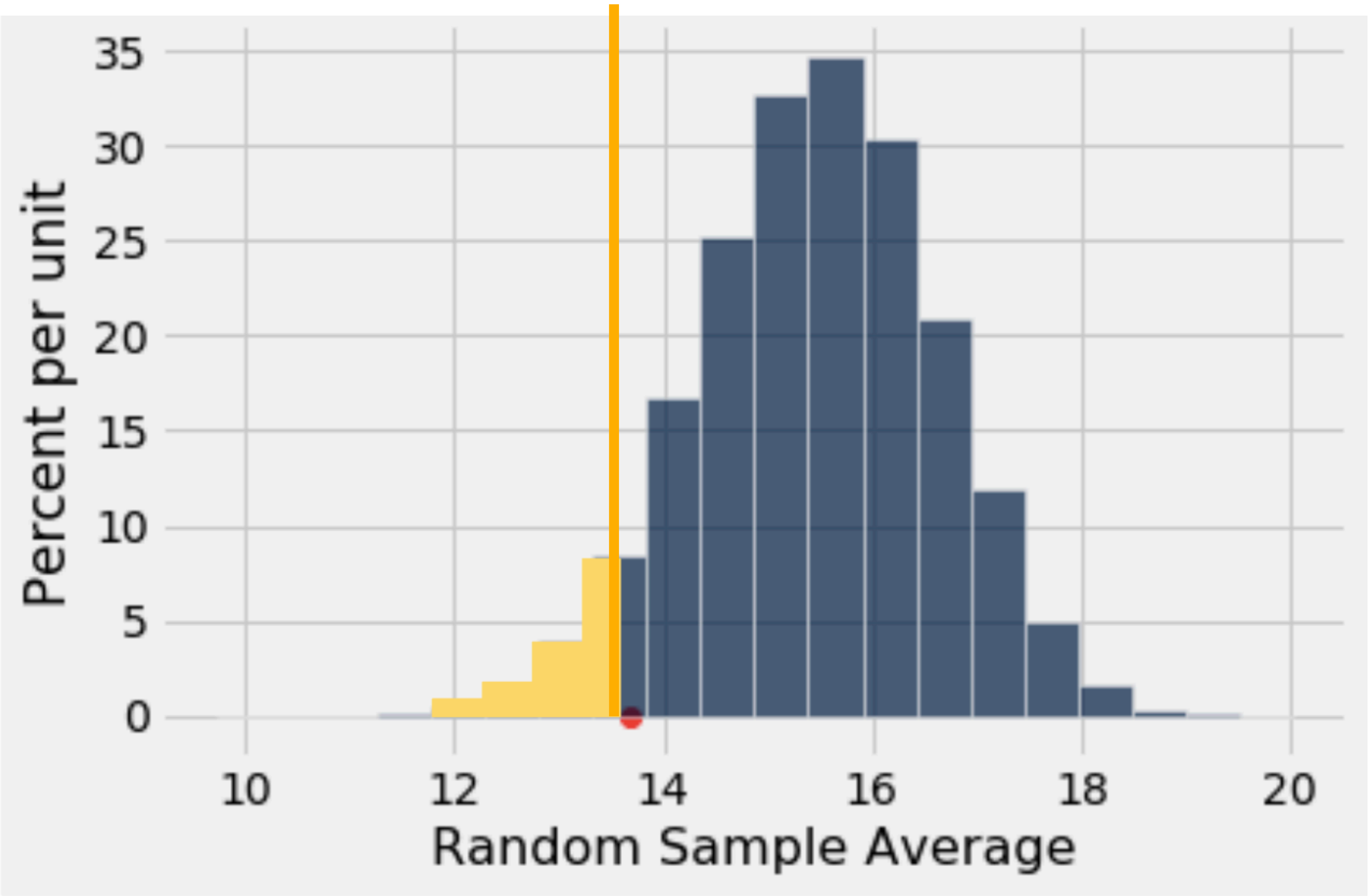


p-value = 0.058

p-value & **statistical significance**

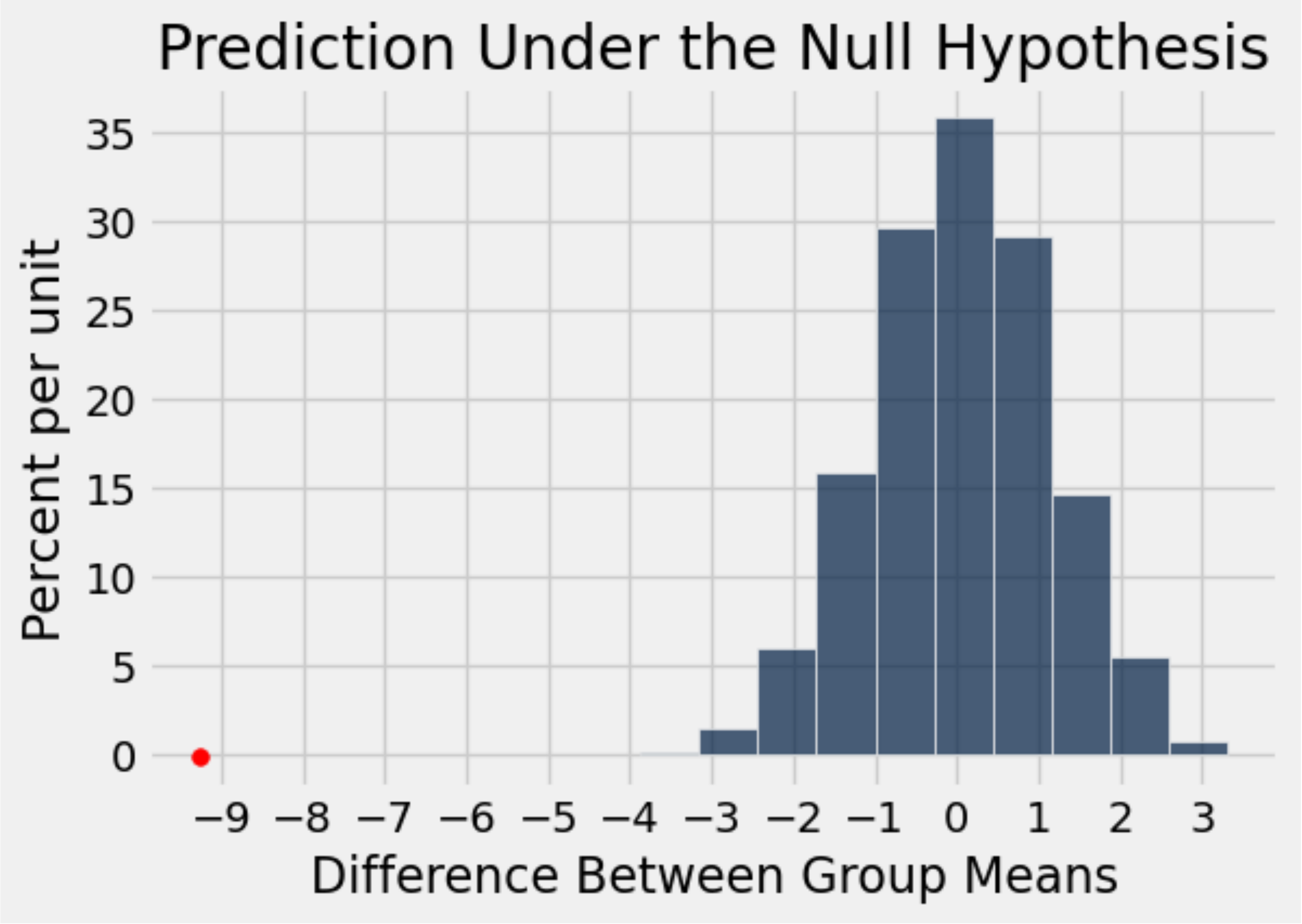
Process:

- Calculate the area of the tail (to the left/right of our observed value)

Hypothesis Testing

- Modeling expected outcomes under the null and comparing it to our observed outcome

Smoking vs Non-Smoking Mothers & Birthweight



p-value = 0

A/B Testing

Compare the difference between two groups

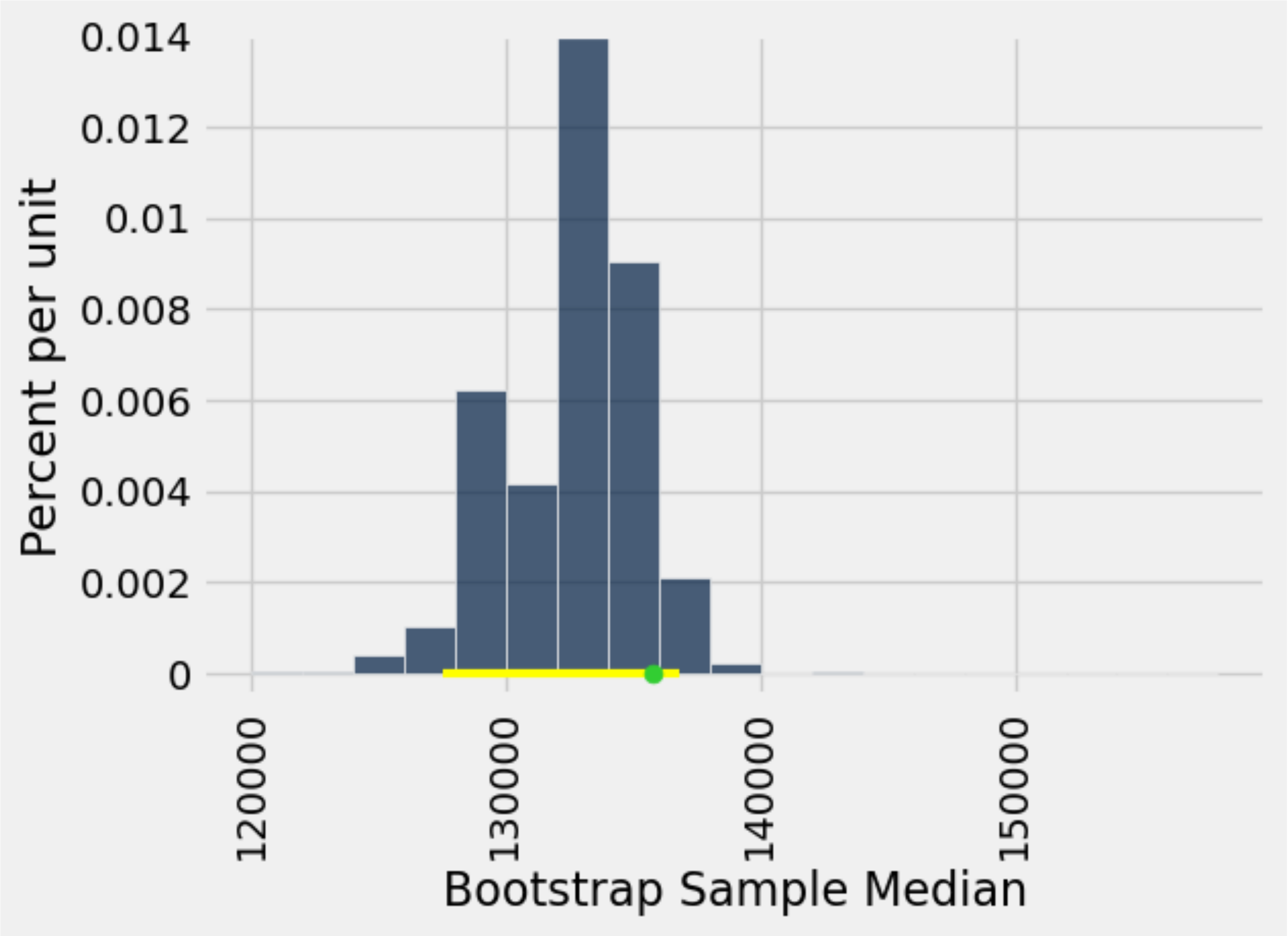
Process:

- Permutation test (shuffle labels)

Estimating a Parameter

- We want to estimate a population parameter from a sample statistic

Median Employee Salary



95% Confidence Interval: Median Salary between $125,745 and $140,318

Confidence Interval

Lets us estimate a range for what we think the parameter's value is

Process:

- Bootstrap

Use Methods Appropriately

When *not* to use the Bootstrap

- If you're trying to estimate very high or very low percentiles
- If you're trying to estimate any parameter that's greatly affected by rare elements of the population (e.g., min or max)
- If the probability distribution of your statistic is not roughly bell shaped
    - The shape of the empirical distribution will be a clue
- The original sample is very small

When to find a confidence interval

- You have to guess a parameter for a population
- You have a random sample from the population
    - But not access to the population
- You want to quantify uncertainty
- A statistic is a reasonable estimate of the parameter

Can you use a confidence interval like this?

Suppose our 95% confidence interval for the average age of mothers is the population is [26.9, 27.6] years

- **True or False**: About 95% of the mothers in the population were between 26.9 years and 27.6 years old.

- **True or False**: There is about 95% probability that the average age of the mothers in the population is in the range 26.9 years to 27.6 years old.

Can you use a confidence interval like this?

Suppose our 95% confidence interval for the average age of mothers is the population is [26.9, 27.6] years

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    - **False**. We are estimating the **average age** is in this interval
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  - **False**. We are estimating the **average age** is in this interval
- **True or False**: There is about 95% probability that the average age of the mothers in the population is in the range 26.9 years to 27.6 years old.
  - **False**. The average age is unknown but **constant**. It is not random.

Average and Histograms

Review: Averages/Means

Suppose we have an array [2, 3, 3, 9].

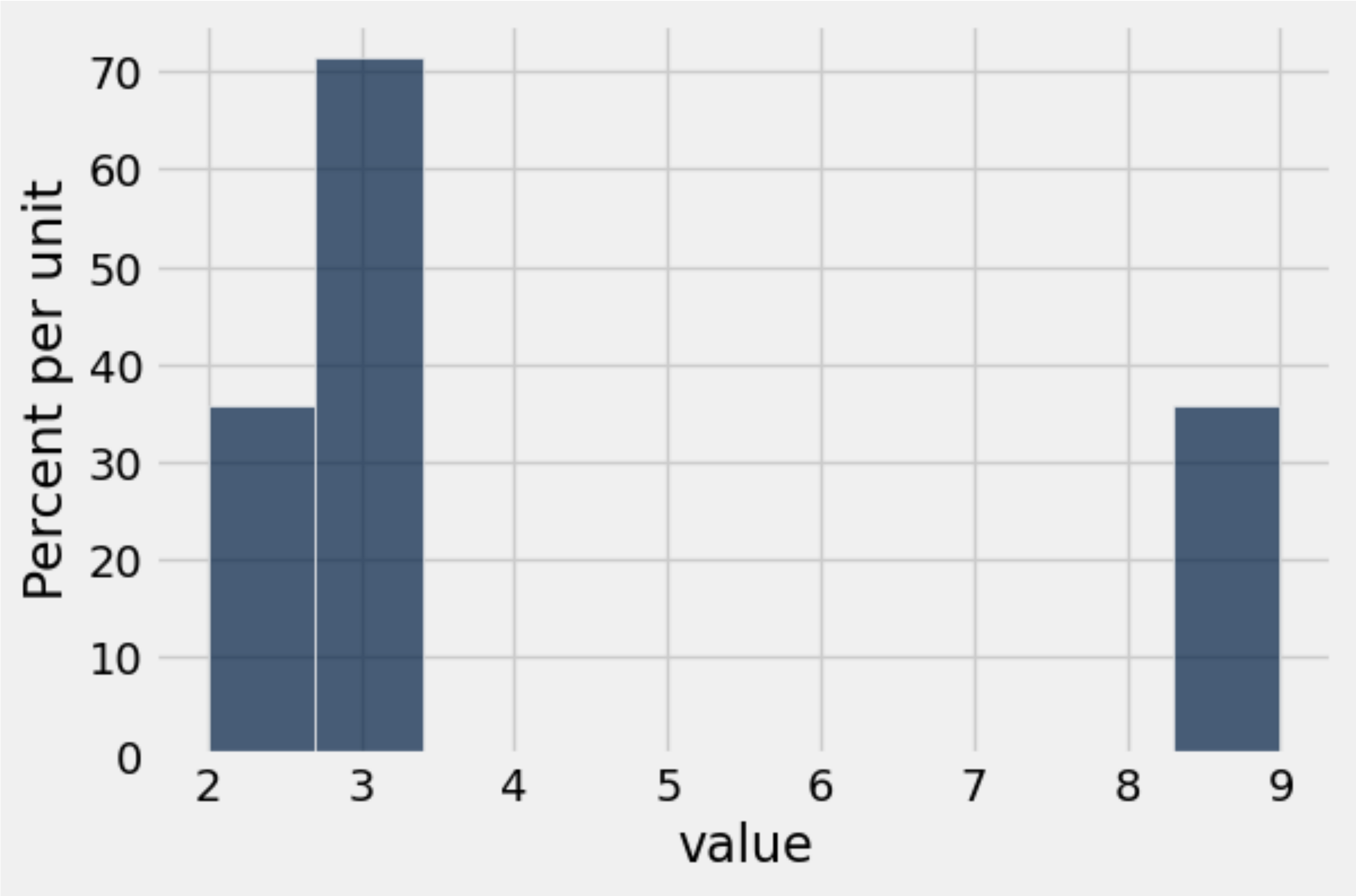
We can compute the average as: $\text{average} = \frac{2+3+3+9}{4} = 4.25$

Notice:

- Need not be a value in the list
- Need not be an integer even if the data consists of integers
- Somewhere between the min and max, but not necessarily the halfway between min and max
- Same units as the data

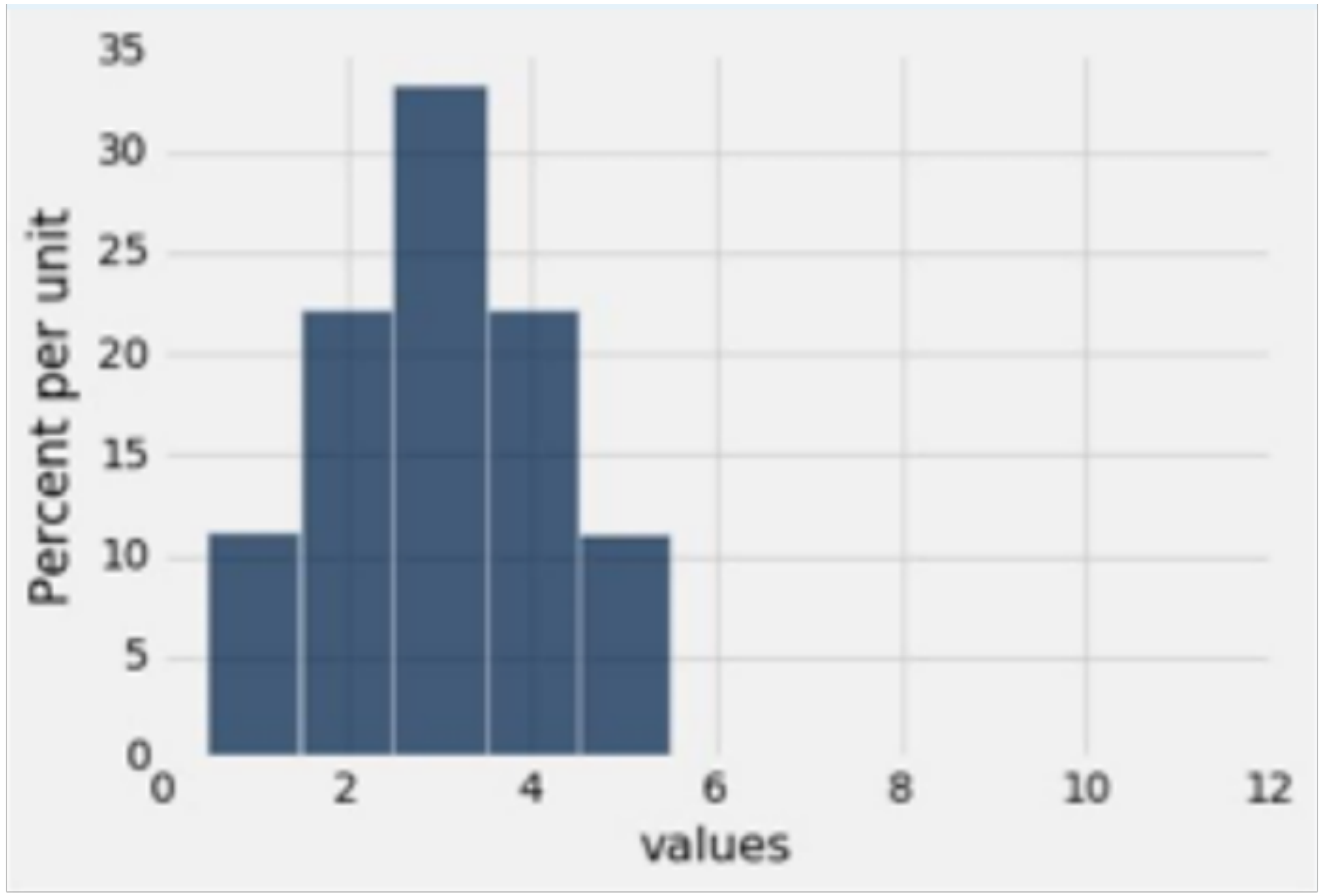
Relation to Histograms

- The average depends only on the **proportions** in which the distinct values appear
- The average is the **center of gravity** of the histogram
- It is the point on the horizontal axis where the histogram balances



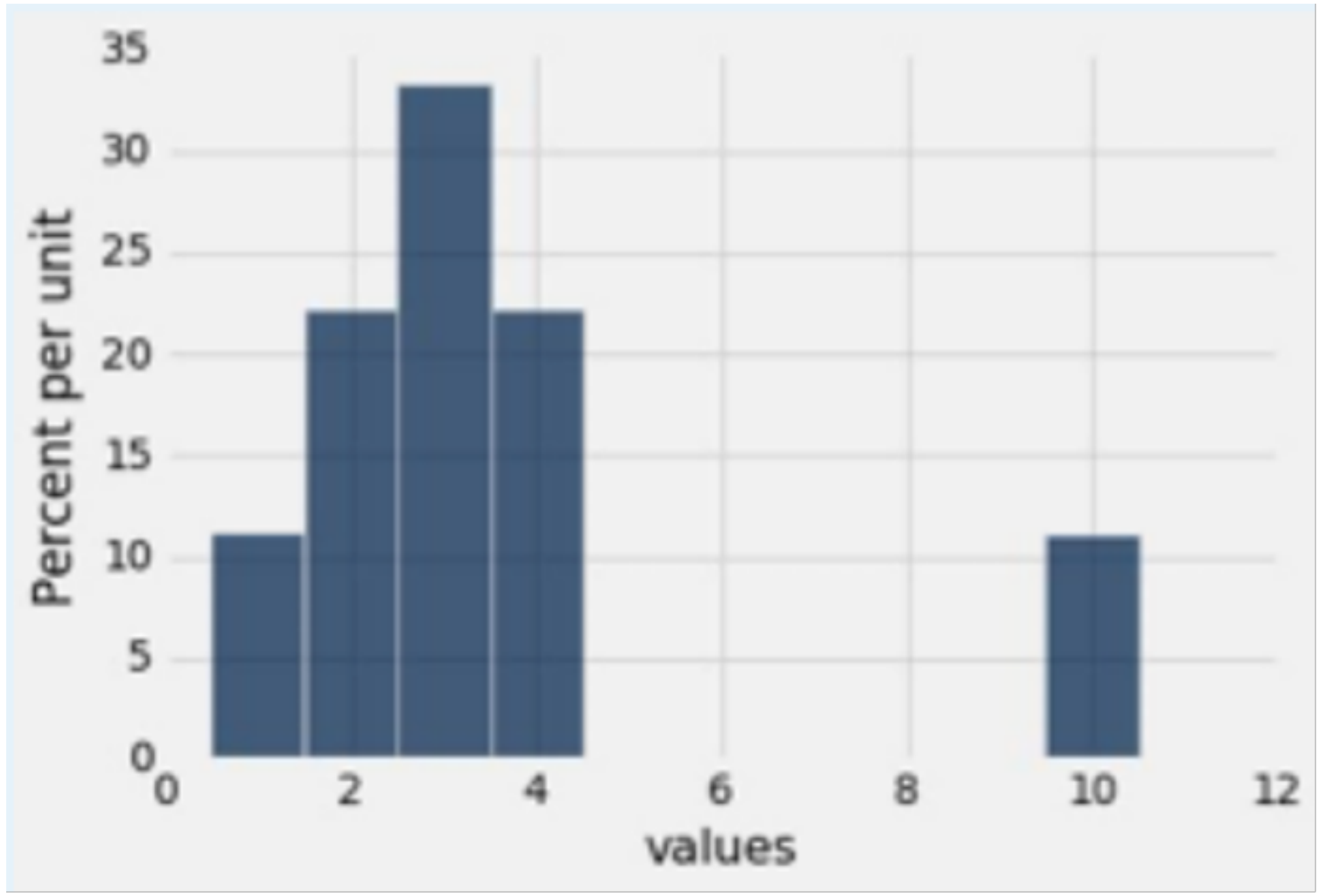
Average and Median

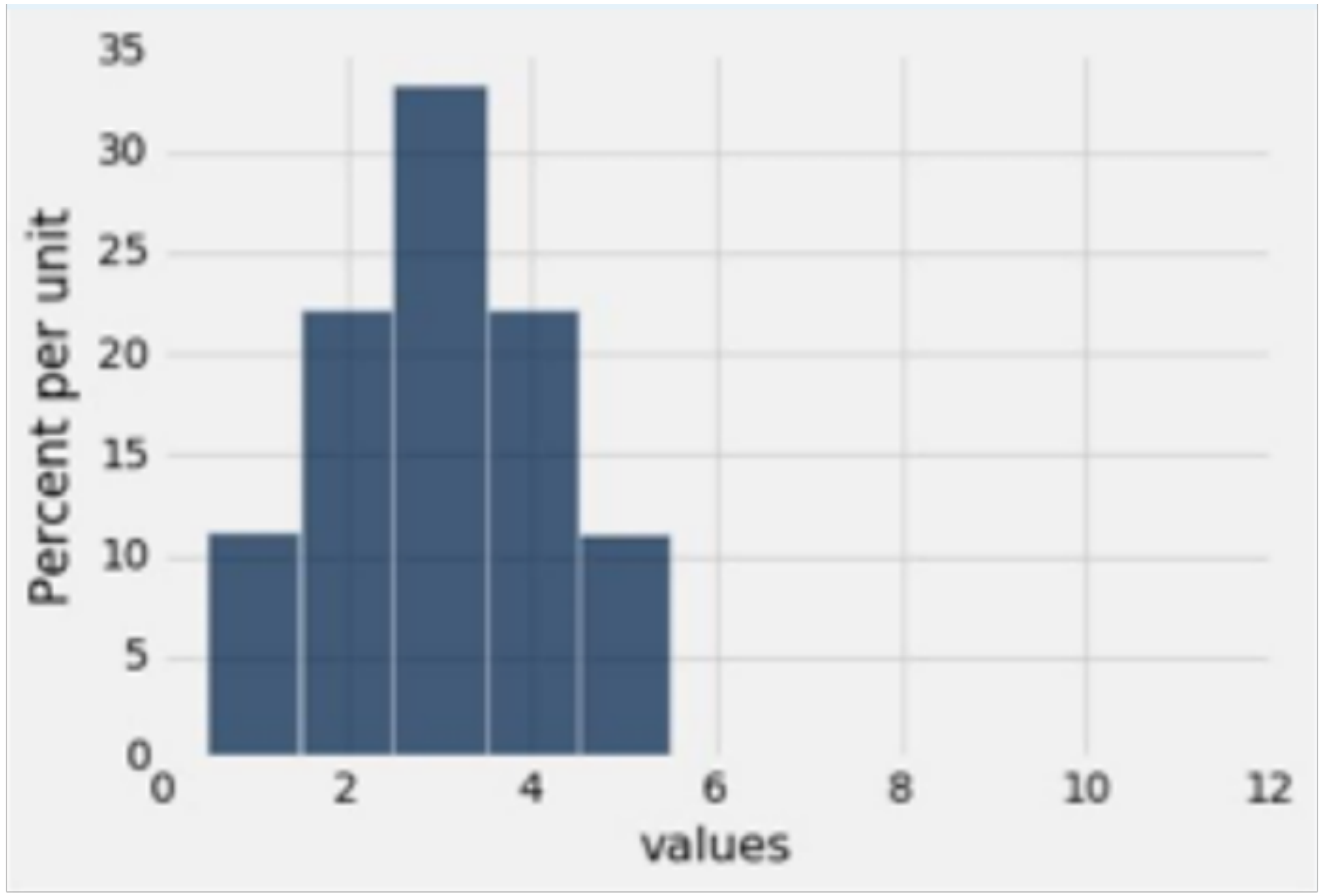
- [1,2,2,3,3,3,4,4,5]
- What is the average?
    - 3
- What is the median?
    - 3



Average and Median

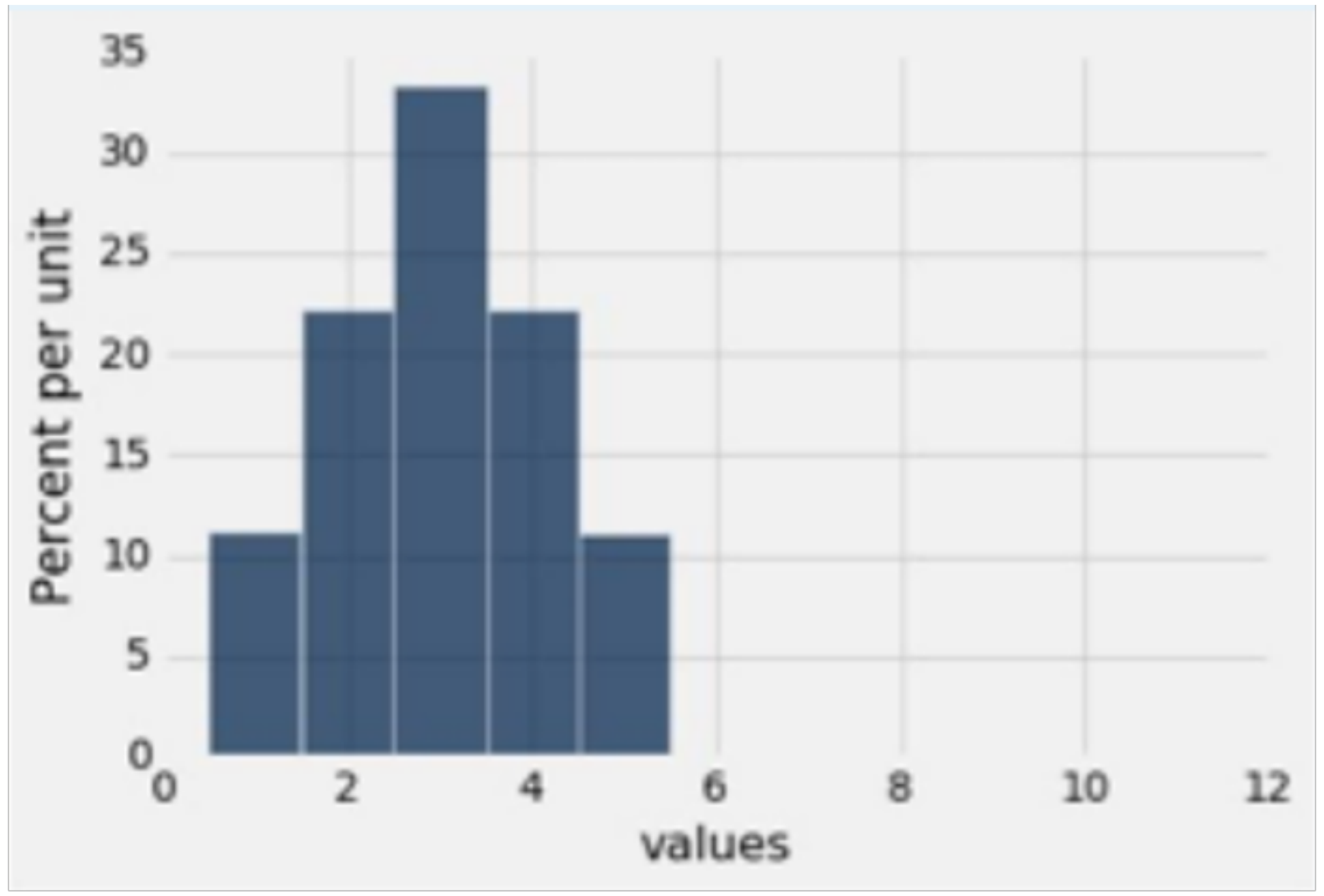
- Are the medians of these two the same or different?
- Are the means the same or different?



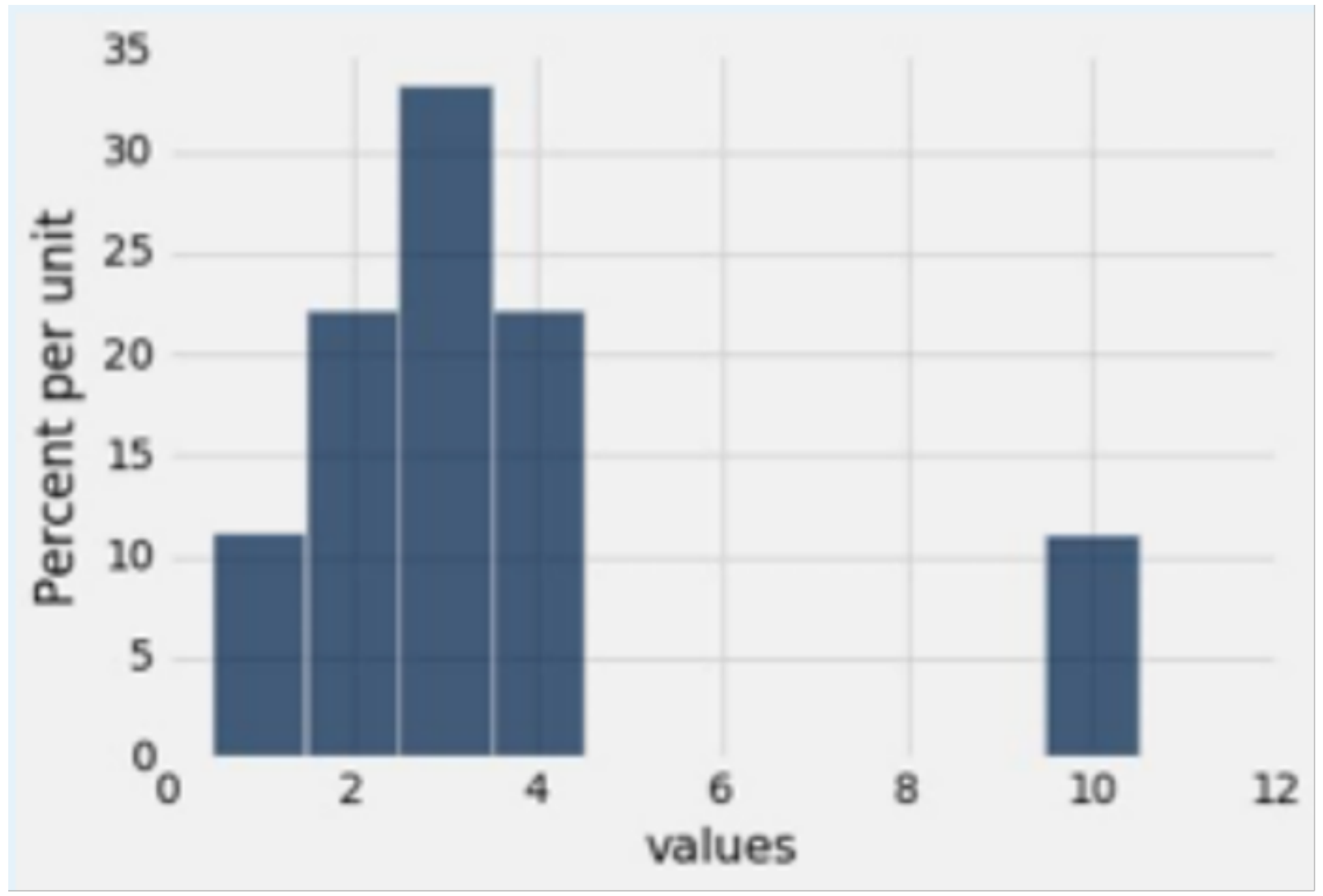


Average and Median

- List 1: [1,2,2,3,3,3,4,4,5]
    - Median =
    - Average =

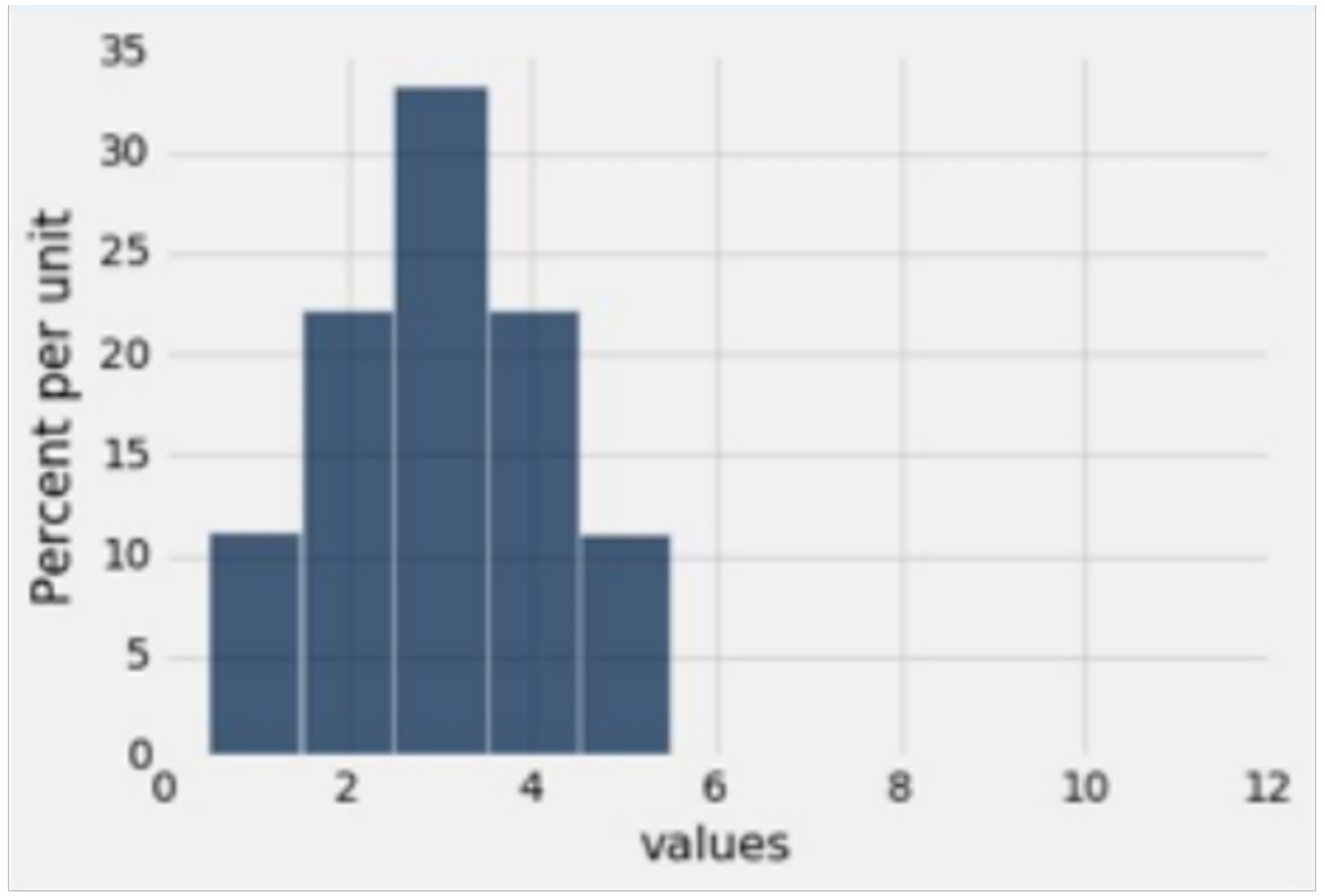


- List 2: [1,2,2,3,3,3,4,4,10]
    - Median =
    - Average =



Average and Median

- List 1: [1,2,2,3,3,3,4,4,5]
    - Median = 3
    - Average =



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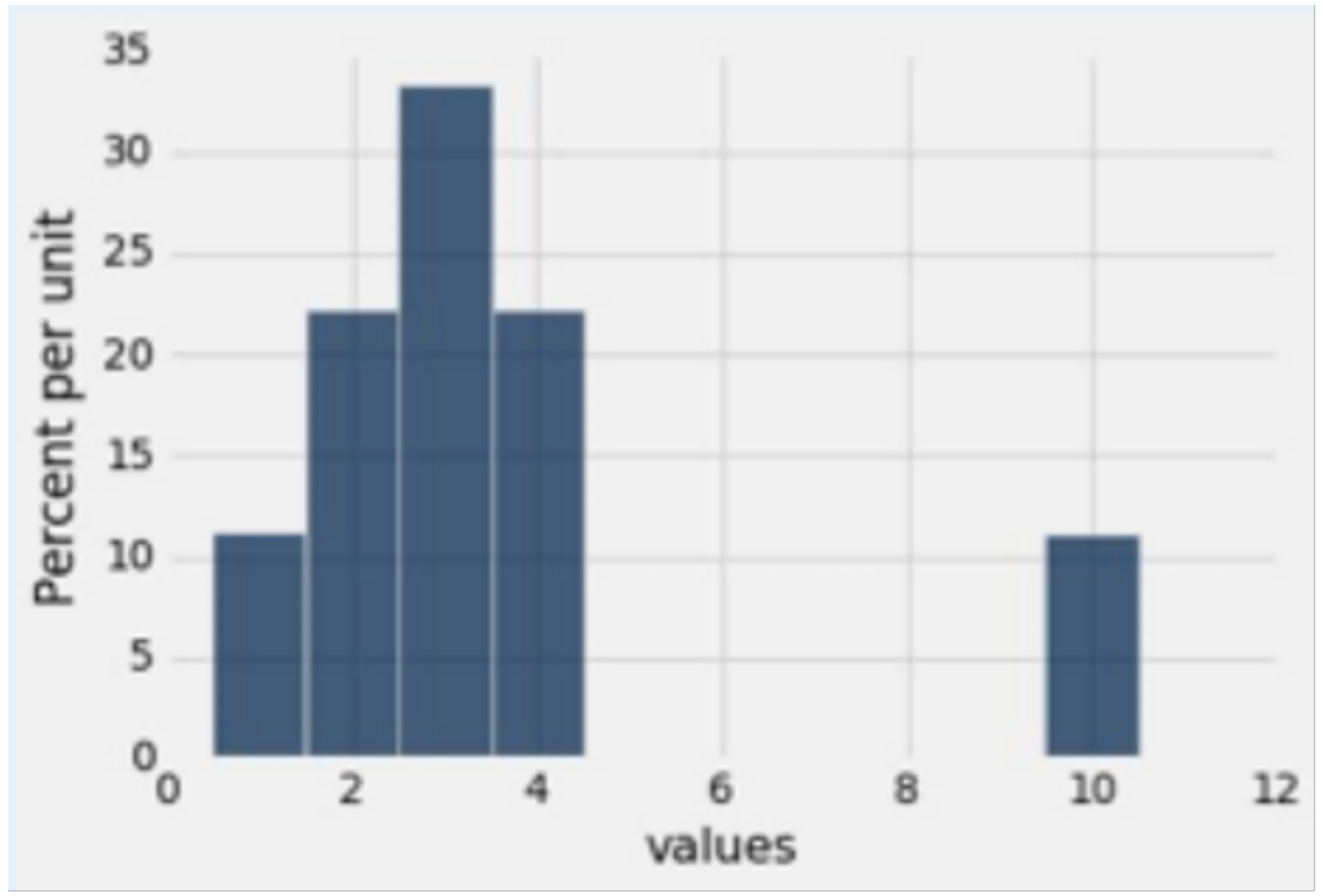

Average and Median

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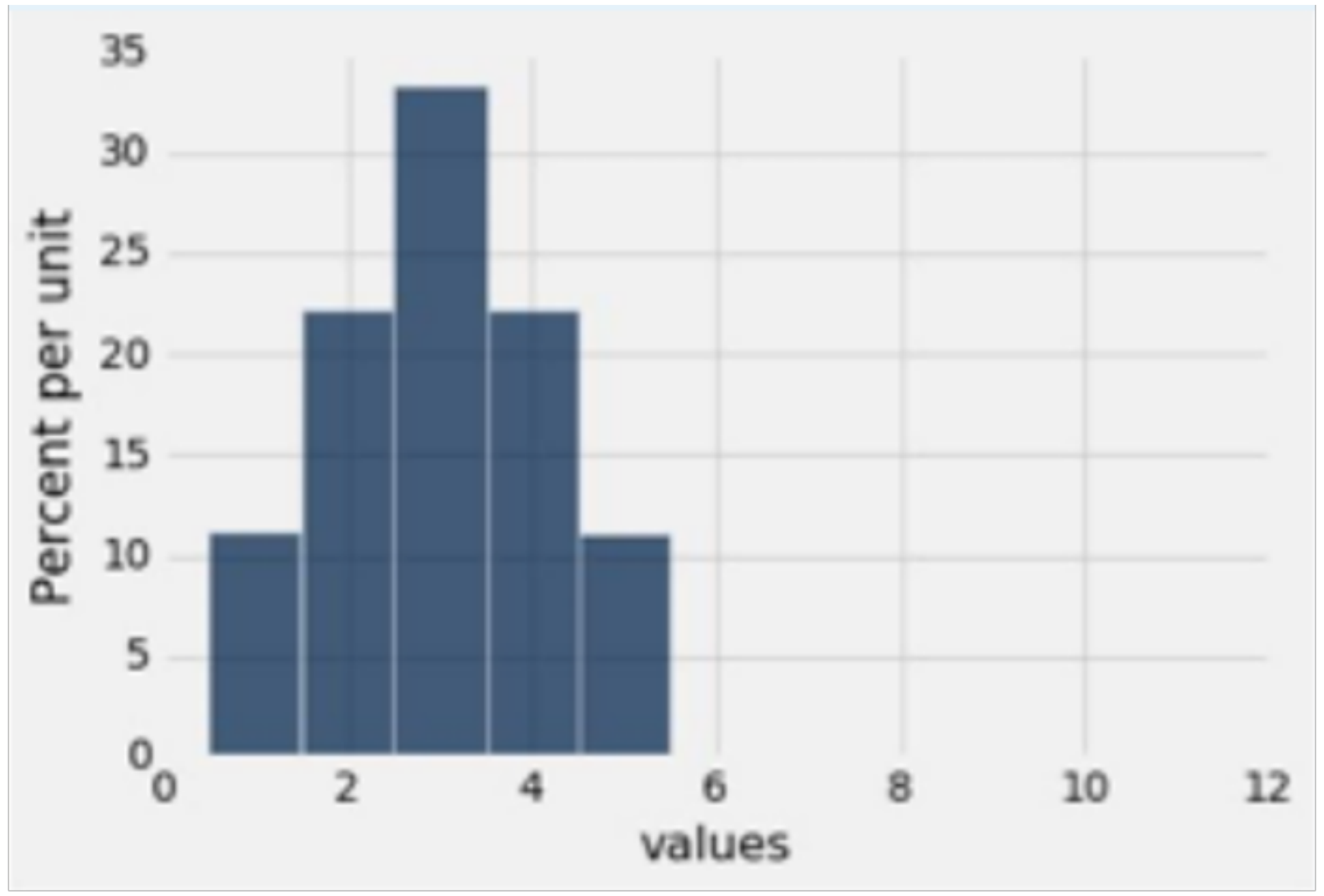
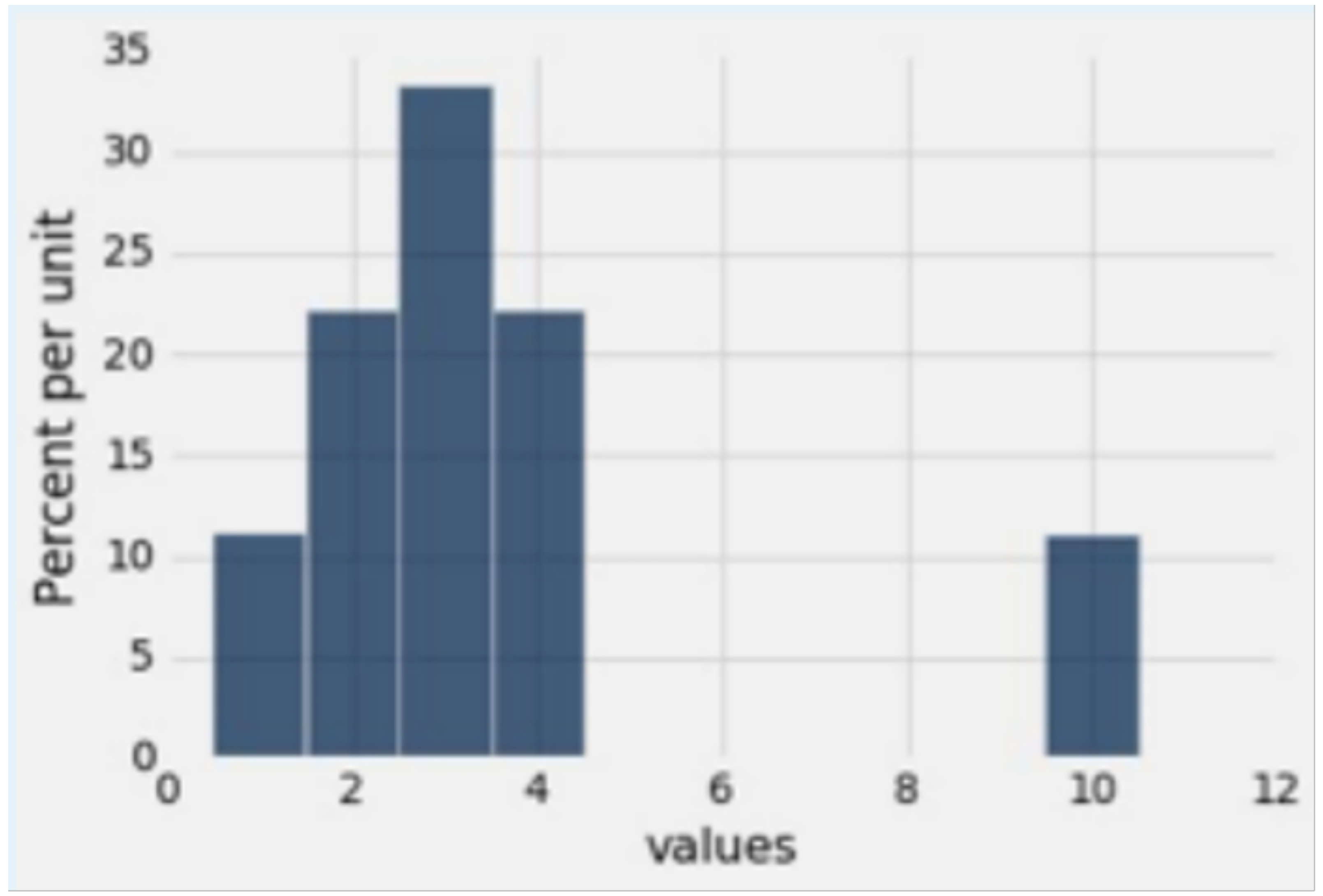


- List 2: [1,2,2,3,3,3,4,4,10]
  - Median = 3
  - Average = 3.55556



Comparing Mean and Median

- **Mean**: Balance point of the histogram
- **Median**: Half-way point of the data. Half of the area of the histogram is on either side of the median
- If the distribution is **symmetric** about a value, then that value is both the average and the median
- If the histogram is **skewed**, then the mean is pulled away from the median in the direction of the tail

Standard Deviation

Variability

- Center of gravity of a histogram is the mean
    - What about the values on either side?
- Variability is how we describe how far apart values are spread away from the center (mean)

Deviation from the Average

We can compute the deviation from the average of a value from this list as:

$$\text{deviation} = \text{value} - \text{mean}$$

- If a value is above the mean, the deviation is **positive**

- If a value is below the mean, the deviation is **negative**

- Deviations tell us the *direction* and *size* of the difference

How can we use this to define variability?

How to Define Variability?

- To measure for how far the numbers are spread from the mean:
  - Compute the average

Let $\overrightarrow{V}$ be a collection of values and $\mu = \text{avg}\left(\overrightarrow{V}\right)$

How to Define Variability?

- To measure for how far the numbers are spread from the mean:
  - Compute the average
  - Compute each value's deviation from the average

Let $\overrightarrow{V}$ be a collection of values and $\mu = \text{avg}\left(\overrightarrow{V}\right)$

$$v - \mu \quad \text{for } v \in \overrightarrow{V}$$

How to Define Variability?

- To measure for how far the numbers are spread from the mean:
  - Compute the average
  - Compute each value's deviation from the average
  - Square the deviations

Let $\overrightarrow{V}$ be a collection of values and $\mu = \text{avg}\left(\overrightarrow{V}\right)$

$$(v - \mu)^2 \text{ for } v \in \overrightarrow{V}$$

How to Define Variability?

- To measure for how far the numbers are spread from the mean:
  - Compute the average
  - Compute each value's deviation from the average
  - Square the deviations
  - Compute the mean of the these squared deviations

Let $\overrightarrow{V}$ be a collection of values and $\mu = \text{avg}\left(\overrightarrow{V}\right)$

$$\text{avg}\left((v - \mu)^2 \text{ for } v \in \overrightarrow{V}\right)$$

How to Define Variability?

- To measure for how far the numbers are spread from the mean:
  - Compute the average
  - Compute each value's deviation from the average
  - Square the deviations
  - Compute the mean of the these squared deviations

Let $\overrightarrow{V}$ be a collection of values and $\mu = \text{avg}\left(\overrightarrow{V}\right)$

Variance of $\overrightarrow{V}$
$= \text{avg}\left((v - \mu)^2 \text{ for } v \in \overrightarrow{V}\right)$

Standard Deviation

- To convert our units back to our original units, we need to take the square root
    - This gives us the **standard deviation**

$$\sigma = \sqrt{\operatorname{avg}\left((v-\mu)^2 \text{ for } v \in \overrightarrow{V}\right)}$$

- To compute the standard deviation of `arr`:
    - `np.std(arr)`

Let $\overrightarrow{V}$ be a collection of values and $\mu = \operatorname{avg}\left(\overrightarrow{V}\right)$

Variance of $\overrightarrow{V}$

$$= \operatorname{avg}\left((v-\mu)^2 \text{ for } v \in \overrightarrow{V}\right)$$

Standard Deviation (SD)

Standard deviation is the root mean square of deviations from the average

$$\sigma = \sqrt{\operatorname{avg}\left((v-\mu)^2 \text{ for } v \in \overrightarrow{V}\right)}$$

Why we like standard deviation:

- No matter the shape of the distribution, the bulk of the data is in the range "average plus or minus a few standard deviations"

- It has a nice relation with the bellcurve (to be discussed later)

Chebyshev's Inequality

Chebyshev's Inequality: *No matter what the shape of the distribution*, the proportion of values in the range "$\text{average} \pm z \text{ SDs}$" is at least $1 - \frac{1}{z^2}$

- Note this is a lower bound, not an exact answer
  - The proportion of entries within the range "$\text{average} \pm z \text{ SDs}$" could be much larger than $1 - \frac{1}{z^2}$, but it can't be smaller

Chebyshev’s Bounds

Range	Proportion
$\text{average} \pm 2\ \text{SDs}$	at least $1 - \frac{1}{4} = 75\,\%$
$\text{average} \pm 3\ \text{SDs}$	at least $1 - \frac{1}{9} \approx 89\,\%$
$\text{average} \pm 4\ \text{SDs}$	at least $1 - \frac{1}{16} = 93.75\,\%$
$\text{average} \pm 5\ \text{SDs}$	at least $1 - \frac{1}{25} = 96\,\%$

True no matter what the distribution looks like

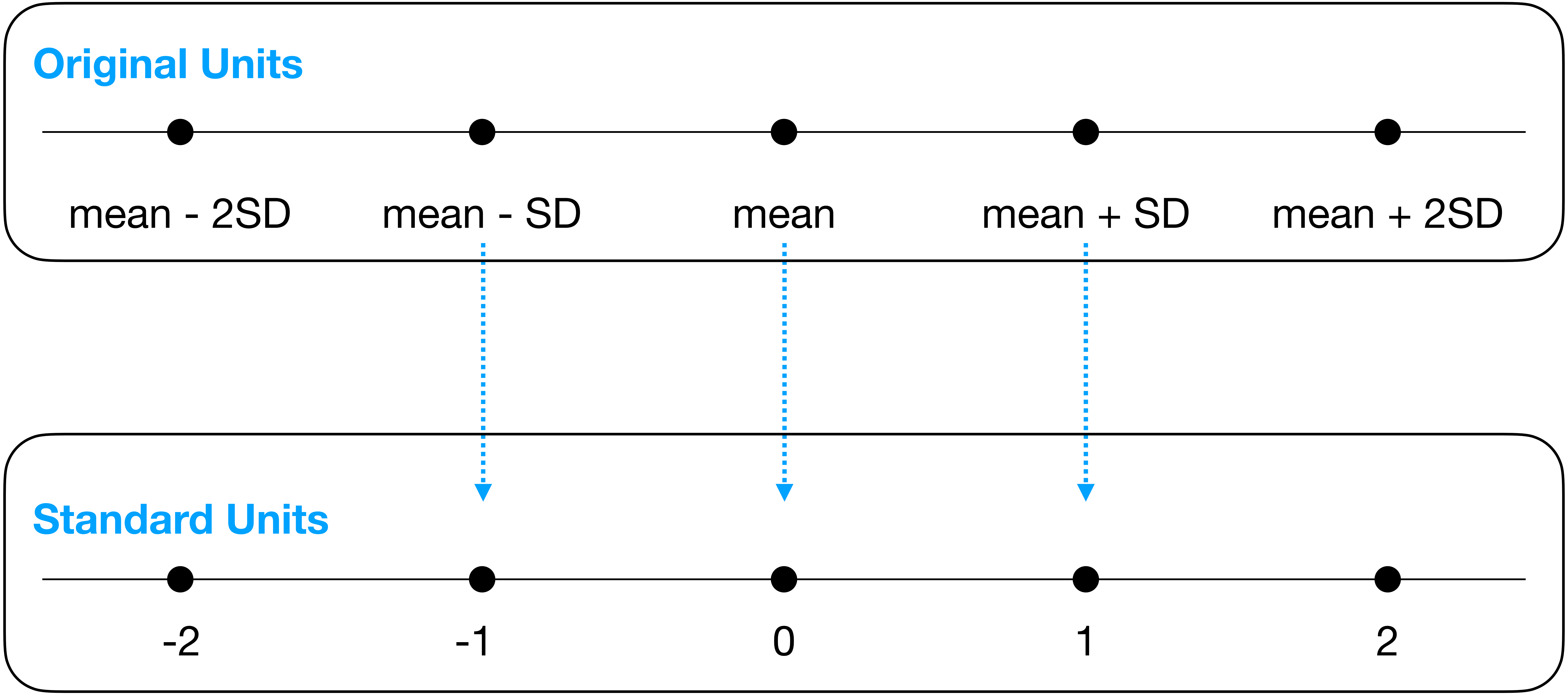
Standard Units

Standard Units

- The quantity z (from "$\text{average} \pm z\ \text{SDs}$" in Chebychev's inequality) measures **standard units**
    - **Standard units** is the number of standard deviations away from the average
- To convert a value (v) to standard units, compare the deviation from the average (μ) with the standard deviation (SD):

$$z = \frac{v - \mu}{\text{SD}}$$

Converting to Standard Units



Interpreting Standard Units

$$z = \frac{v - \mu}{\text{SD}}$$

- When z is **negative**, the value v is **below** average

- When z is **positive**, the value v is **above** average

- When z is **0**, the value v **is** the average

When values are in standard units, average = 0, SD = 1

Example

What whole numbers are closest to:

- Average age?

- The SD of ages?

Age in Years	Age in Standard Units
27	-0.0392546
33	0.992496
28	0.132704
23	-0.727088
25	-0.383171
33	0.992496
23	-0.727088
25	-0.383171
30	0.476621
27	-0.0392546

Example

What whole numbers are closest to:

- Average age?
    - **27**. The standard unit is close to 0
- The SD of ages?

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Example

What whole numbers are closest to:

- Average age?
  - **27**. The standard unit is close to 0
- The SD of ages?
  - About **6** years. The standard unit at 33 is close to 1 and $33 - 27 = 6$

Age in Years	Age in Standard Units
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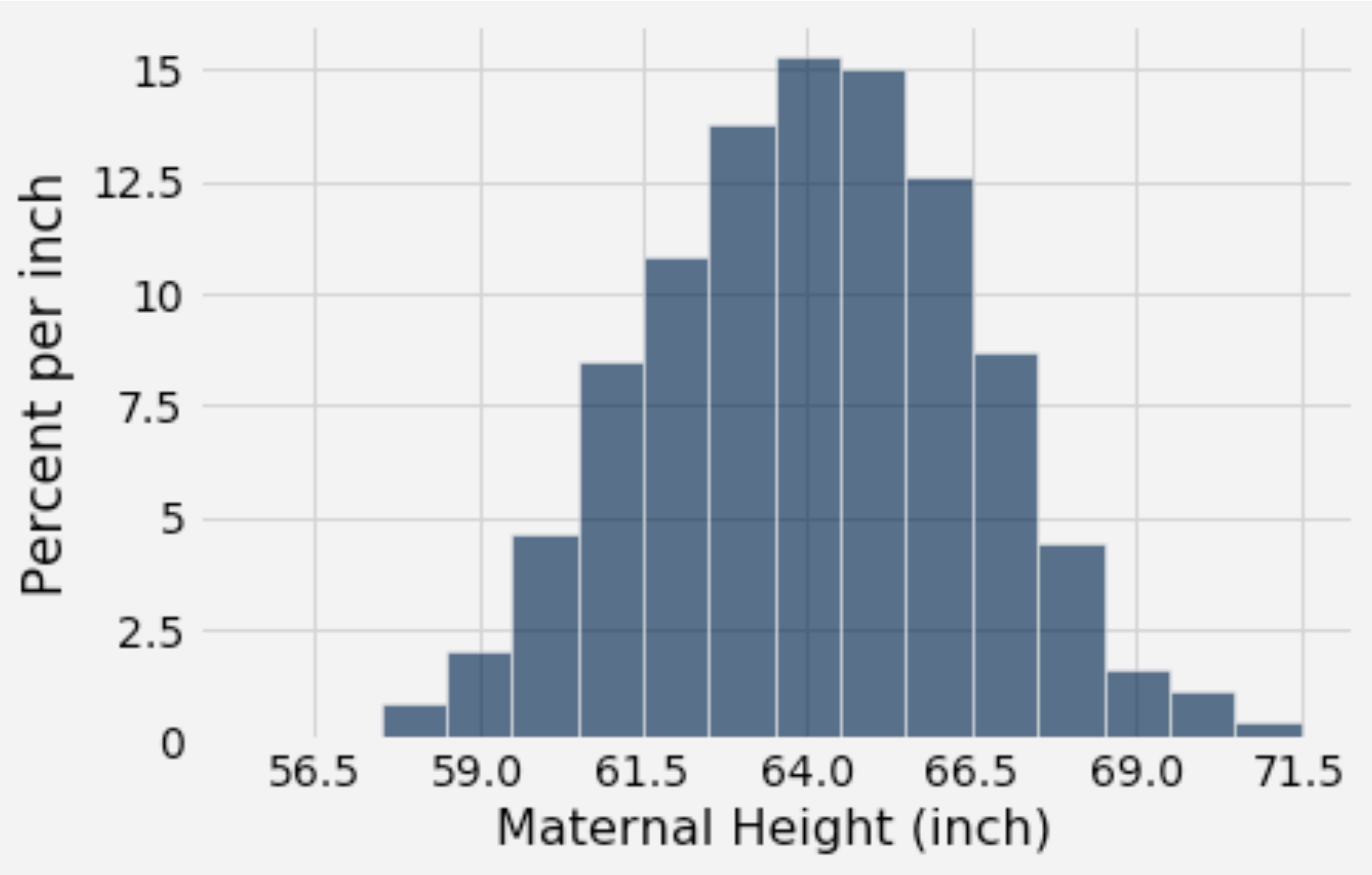
27	-0.0392546

Normal Distribution

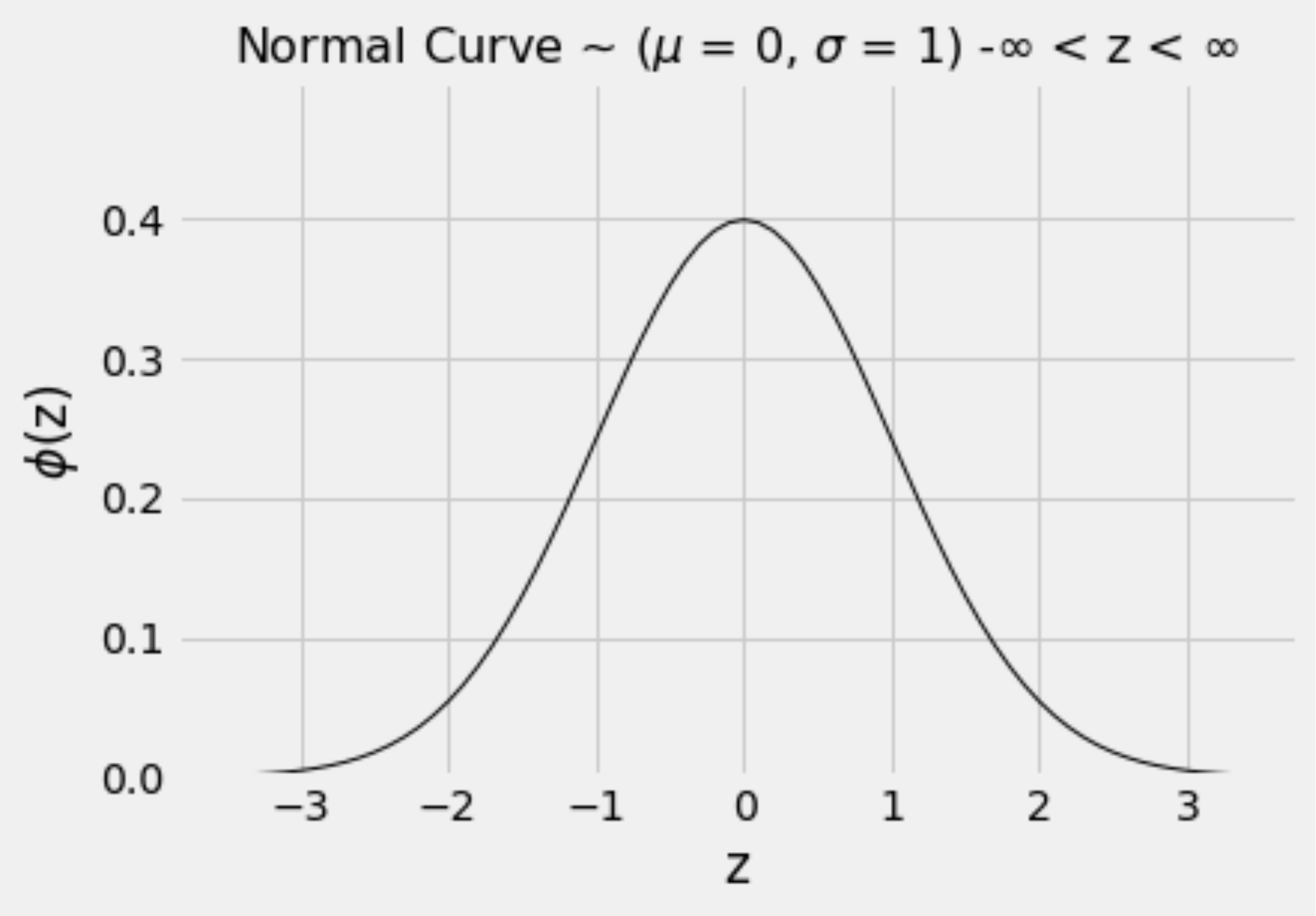
Bell Shaped Curves

- The normal curve / bell-curve is a very common distribution
- For bell-shaped (aka Gaussian distribution):
  - Average is at the center
  - SD is the distance between the average and the points of inflection on either side



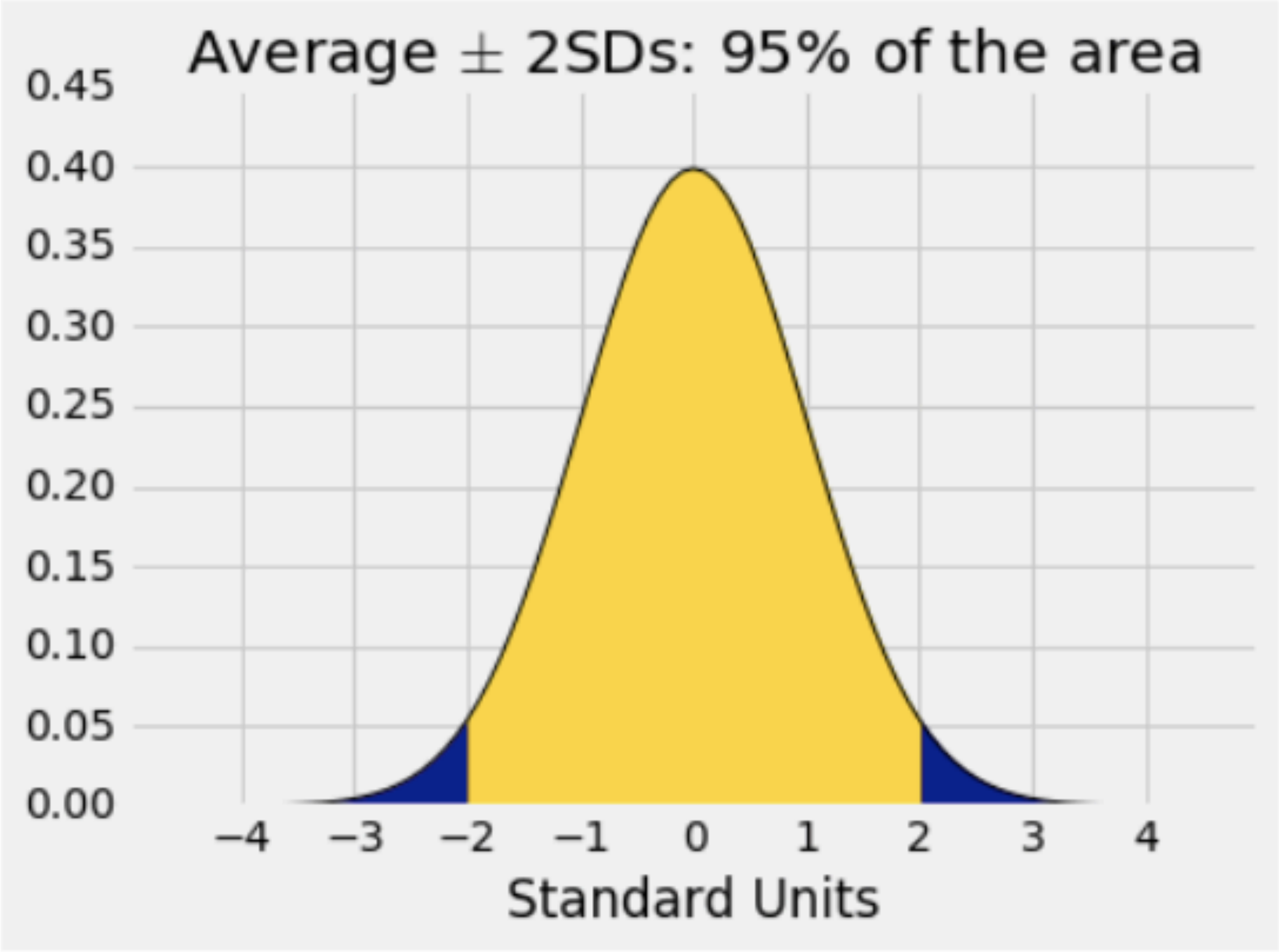
Normal Distribution

- On a standard normal curve, x-axis units are standard units
- Total area of the curve is 1
- Curve is symmetric around 0 (mean and median are both 0)
- Points of inflection are -1 and 1
  - Standard deviation is 1



Application to Normal Distributions

- If a histogram is bell-shaped (normal), then 95% of the data is in the range $\text{average} + 2\ \text{SDs}$
- Note this is much higher than Chebychev's bound of 75%
    - 75% is a lower bound that applies to *all* distributions



Normal vs All Distributions

Range	All Distributions (Chebyshev's)	Normal Distribution
mean $\pm$ 1 SDs	At least 0%	At least 68%
mean $\pm$ 2 SDs	At least 75%	At least 95%
mean $\pm$ 3 SDs	At least 89%	At least 99%

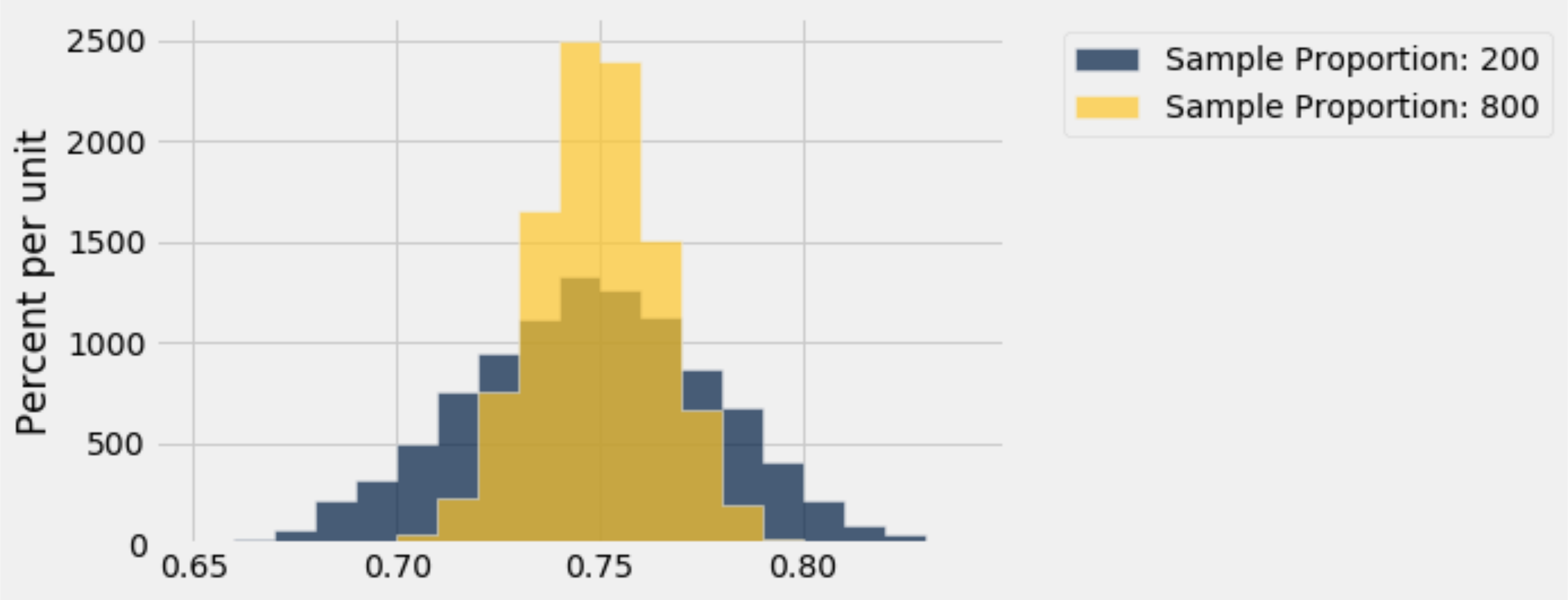
Central Limit Theorem

- Describes how a normal distribution is connected to random sample averages (which helps us determine the population average)

- **Central Limit Theorem**: If a sample is large and drawn at random with replacement, then regardless of the distribution the **probability distribution of the sample average** is roughly normal

Central Limit Theorem

- Next time: how can we use this property to help us determine the sample size we need to draw useful conclusions?



Next time

- Central Limit Theorem